ELECTROMAGNETIC FIELDS IN PERIODIC LINEAR TRAVELLING-WAVE STRUCTURES

J.P. Pruiksma, R.W. de Leeuw, J.I.M. Botman*, H.L. Hagedoorn, Cyclotron Laboratory
A.G. Tijhuis, Electromagnetics Department,
Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, the Netherlands

Abstract

An analytical description of the electromagnetic field in a periodically disk-loaded circular waveguide is given. The field is expressed in terms of the waveguide modes. The main advantage of this approach is that each mode matches the boundary conditions in the empty waveguide. These modes have convenient orhogonality properties. First, a single diaphragm in the waveguide is considered and the reflection problem arising from one incident waveguide mode is solved with the mode-matching technique. Then a matrix eigenvalue equation is derived for the periodically loaded waveguide. The solution of this equation yields the dispersion curve for the structure and leads to the full field description for a given operating mode of the accelerator.

TM$_{0n}$ modes of a circular waveguide

A circular waveguide of radius b, centered around the z-axis is considered. For the acceleration of particles in the disk-loaded waveguide, only the transverse magnetic (TM) modes of the electromagnetic field are of interest. A time dependence of $e^{i\omega t}$ and a z dependence of $e^{\Gamma_n z}$ is assumed and substituted into Maxwell's equations. The axially symmetric TM_{0n} mode solutions are:

$$e_{zn} = \pm\frac{\sqrt{2}\alpha_n}{b^2\Gamma_n J_1(\alpha_n)}J_0(\frac{\alpha_n}{b}r)e^{\mp\Gamma_n z}, \quad (1)$$

$$e_{rn} = \frac{\sqrt{2}}{bJ_1(\alpha_n)}J_1(\frac{\alpha_n}{b}r)e^{\mp\Gamma_n z} = \phi_n e^{\mp\Gamma_n z}, \quad (2)$$

$$h_{\varphi n} = \pm Y_n\phi_n e^{\mp\Gamma_n z}, \quad (3)$$

$$\frac{\alpha_n^2}{b^2} = \Gamma_n^2 + \frac{\omega^2}{c^2}. \quad (4)$$

For the nth mode, the z-component of the electric field is e_{zn}, the radial electric field- and azimuthal magnetic field components are e_{rn} and $h_{\varphi n}$ respectively. The wave admittance $Y_n = \frac{i\epsilon_0\omega}{\Gamma_n}$ and α_n is the nth root of the Bessel function $J_0(x)$. The functions ϕ_n defined in equation (2) are orthonormal:

$$\int_0^b \phi_n\phi_m r dr = \delta_{nm}. \quad (5)$$

The mode-matching technique discussed in the next section makes use of this orthonormality. For linear travelling wave accelerating structures it is customary to choose the radius b and the frequency ω in such a way that only the propagation constant Γ_1 is imaginary. All other $\{\Gamma_n\}$ are real and represent attenuating modes.

* Corresponding author

Reflection from a single diaphragm

In the circular waveguide of radius b, an infinitely thin diaphragm with a circular aperture of radius a is placed at $z = 0$, see Fig. 1.

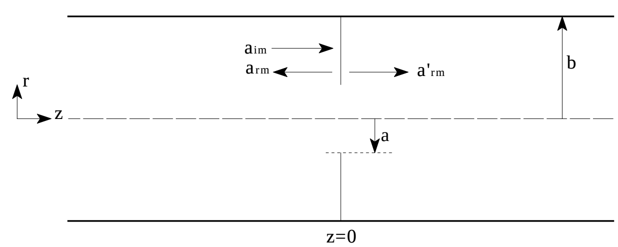

Figure 1: reflection at a diaphragm

The coefficients of the incident modes are a_{im} so a general incident field is given by:

$$\sum_{m=1}^{\infty} a_{im}\phi_m e^{-\Gamma_m z}. \quad (6)$$

Here, the reflection problem is solved for one incident propagating mode: $a_{i1} = 1$ and all other a_{im} are zero. At the diaphragm, there will be an infinite number of reflected and transmitted modes with coefficients a_{rm} and a'_{rm} respectively, because at $z = 0$ a linear combination of all the modes is needed to satisfy the boundary conditions at the diaphragm. The total radial electric field E_r and azimuthal magnetic field H_φ are:
For $z < 0$:

$$E_r = \phi_1 e^{-\Gamma_1 z} + \sum_{m=1}^{\infty} a_{rm}\phi_m e^{\Gamma_m z}, \quad (7)$$

$$H_\varphi = Y_1\phi_1 e^{-\Gamma_1 z} - \sum_{m=1}^{\infty} Y_m a_{rm}\phi_m e^{\Gamma_m z}. \quad (8)$$

For $z > 0$:

$$E'_r = \sum_{m=1}^{\infty} a'_{rm}\phi_m e^{-\Gamma_m z}, \quad (9)$$

$$H'_\varphi = \sum_{m=1}^{\infty} Y_m a'_{rm}\phi_m e^{-\Gamma_m z}. \quad (10)$$

By using the boundary condition $E_r = E'_r = 0$ at the diaphragm for $a < r < b$ and the continuity of the tangential field components in the aperture ($z = 0$): $E_r = E'_r$ and $H_\varphi = H'_\varphi$

for $0 \le r \le a$, a matrix equation can be derived, whose solution yields the coefficients a_{rm} and a'_{rm}. In the derivation, the orthonormality of the ϕ_n functions is used. This procedure is known as the mode-matching technique, see Masterman [1].
To obtain a matrix equation of finite size, the series of reflected and transmitted modes have to be truncated; therefore only a finite number of coefficients are calculated. Once the coefficients a_{rm} and a'_{rm} are found, the total field can be calculated at every position in the waveguide. The most important coefficients are a_{r1} and a'_{r1}. These are better known as the reflection coefficient R and transmission coefficient T. The coefficients are in general complex numbers, and as a measure for R, the susceptance B is defined as:

$$B = \frac{2iR}{1+R}. \tag{11}$$

B is a real-valued quantity [2]. The susceptance B has been calculated as a function of the frequency ω, see Fig. 2. The solid

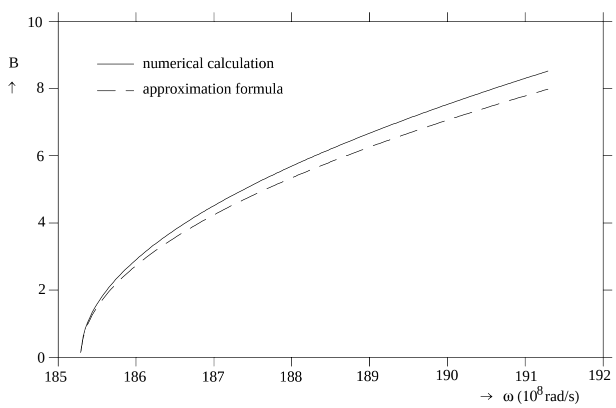

Figure 2: The susceptance as a function of the frequency, using $b = 39$ mm and $a = 10$ mm.

line is calculated by using the mode-matching technique and the dashed line represents an approximation for the susceptance given by an analytical formula derived with the small-aperture theory [3]:

$$B = \frac{3\pi J_1(\alpha_1)}{2\alpha_1^2}\frac{b^4 k}{a^3}, \tag{12}$$

where $ik = \Gamma_1$. This formula was derived by assuming that the aperture diameter is small compared to the guide wavelength $\lambda_g = \frac{2\pi}{k}$. Calculations for smaller aperture radii show an even better agreement between the mode-matching solution and the approximation formula [2].

The periodic structure

In Fig. 3, a section of an infinitely long periodic structure is shown. The structure consists of an empty waveguide with radius b, containing diaphragms with aperture radius a, equally spaced at a distance d. It is assumed that the decaying modes excited at the diaphragms decrease to a negligible value at the neighbouring diaphragms and that only the reflected and transmitted propagating mode is of importance [2]. Once the coefficients for the back and forth propagating modes are found, the coefficients of the decaying modes can be calculated from the single-diaphragm theory discussed in the previous section.

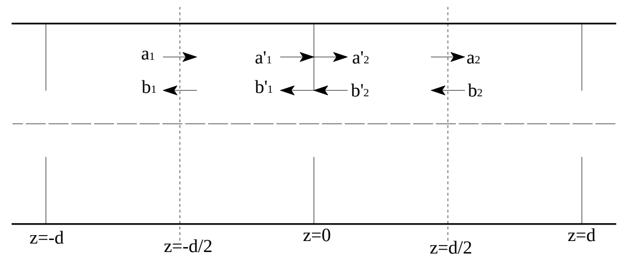

Figure 3: A section of the infinitely long periodic strucure.

The radial electric field of the propagating modes is:
For $-d < z < 0$:

$$E_r^1 = a'_1 \phi_1 e^{-ikz} + b'_1 \phi_1 e^{ikz}. \tag{13}$$

For $0 < z < d$:

$$E_r^2 = a'_2 \phi_1 e^{-ikz} + b'_2 \phi_1 e^{ikz}. \tag{14}$$

Similar equations can be found for the azimuthal magnetic field. In the expression for E_r^1, the term $a'_1\phi_1 e^{-ikz}$ represents the field of the mode propagating in the positive z-direction. When $a'_1 e^{-ikz}$ is seen as an effective coefficient for this mode, the coefficient at the diaphragm ($z = 0$) is a'_1, see Fig. 3. The coefficient at $z = -\frac{d}{2}$ is called a_1 and is given by:

$$a_1 = a'_1 e^{ik\frac{d}{2}}. \tag{15}$$

The other coefficients are defined in a similar way. The coefficients a'_1 and b'_1 are linked to a'_2 and b'_2 in the following way:

$$a'_2 = Rb'_2 + Ta'_1, \tag{16}$$
$$b'_1 = Ra'_1 + Tb'_2. \tag{17}$$

By using equations (16) and (17) together with equation (15) and similar equations for the other coefficients, a transfer matrix can be found which connects the coefficients a_1 and b_1 at $z = -\frac{d}{2}$ with the coefficients a_2 and b_2 at $z = \frac{d}{2}$:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{T}e^{ikd} & -\frac{R}{T} \\ \frac{R}{T} & (T - \frac{R^2}{T})e^{-ikd} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}. \tag{18}$$

The Floquet theorem, see Collin [3], links the fields at position $z = -\frac{d}{2}$ to the fields at position $z = \frac{d}{2}$:

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = e^{-i\beta d} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \tag{19}$$

where $\beta d = \phi$ is the phase shift per cell. With this equation, a phase velocity can be defined, because at the time $\omega t = \beta d$ the fields at $z+d$ are the same as the fields at position z for $t=0$. This gives a phase-velocity:

$$v_p = \frac{\omega}{d}. \tag{20}$$

By combining equations (18) and (19), a matrix eigenvalue equation can be derived, which has the characteristic equation:

$$\cos\beta d = \cos kd - \frac{B}{2}\sin kd. \tag{21}$$

With B the susceptance. From equation (21) it can be observed that the phase shift per cell $\phi=\beta d$ can also be negative, which yields a solution for waves travelling in the negative z-direction. Since B has been calculated as a function of ω and k is also known as a function of ω from equation (4), the phase shift per cell ϕ can be calculated as a function of ω, see Fig. 4. This figure was made using the parameters of the periodic structure of a 10 MeV linear travelling-wave electron accelerator with an operation mode $\phi=\frac{2}{3}\pi$. From Fig. 4, the frequency of this $\frac{2}{3}\pi$ mode can be deduced. Once the frequency has been found, the eigenvalue problem can be solved and the coefficients of the propagating modes are obtained. With these, the coefficients of the decaying modes can be calculated by using the single diaphragm theory. For a phase shift of $\frac{2}{3}\pi$ per cell, three cells are needed for the field description. Figure 5 shows the total longitudinal electric field on the z-axis in the three cells. The dashed line represents the field calculated from the Fourier coefficients of the $\frac{2}{3}\pi$ mode given by the computercode Superfish [4].

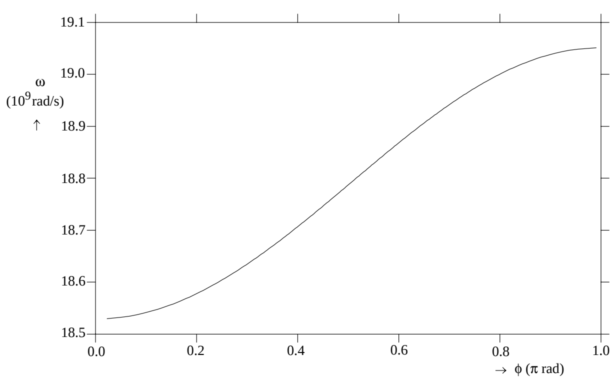

Figure 4: The frequency ω as a function of the phase shift ϕ per cell, using $a\approx 10$ mm, $b\approx 39$ mm and $d\approx 33.33$ mm.

Concluding remarks

The empty waveguide modes are a useful tool for the description of the electromagnetic field in periodically disk-loaded waveguides. With the mode-matching technique, the reflection of waves from an infinitely thin diaphragm is described accurately. The dispersion curve of the infinitely long periodic structure can be calculated and the calculated fields for a given frequency ω agree reasonably well with the fields calulated by Superfish.

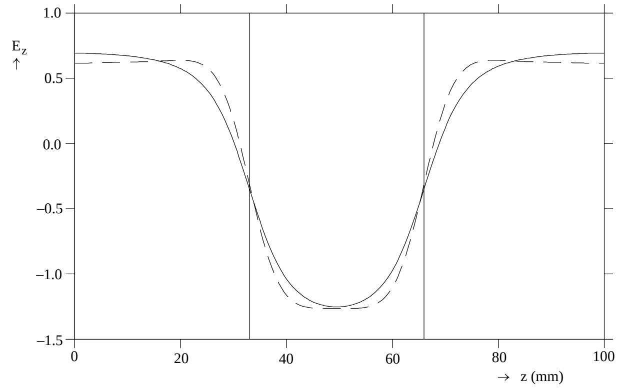

Figure 5: The E_z-field on the axis for the $\frac{2}{3}\pi$ mode. The solid line is the field calculated with the theory and the dashed line represents the field calculated with Superfish.

To obtain more accurate results, the theory could be extended to include diaphragms of finite thickness [2] [5] and also to a description of aperiodic structures [6], which is important for the design of low-energy travelling-wave linacs.

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